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The number of homomorphisms from a finite abelian group to a finite group

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1 Generating functions

Let A be a finitely generated group, and let $\mathcal{F}_A$ be the set of all subgroups B such that the factor groups A/B are finite groups. Let G be a finite group, and let S_n be the symmetric group on n letters. For the wreath product $G \wr S_n$, put

$$h_n(A;G) = \begin{cases} |\mathrm{Hom}(A, G \wr S_n)| & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

For each subgroup B of A, let $h(B,G) = |\mathrm{Hom}(B,G)|$.

Theorem 1. 1 *Let A be a finitely generated group, and let G be a finite group. Then,*

$$1 + \sum_{n=1}^{\infty} \frac{h_n(A;G)}{|G|^n n!} \cdot x^n = \exp\left(\sum_{B\in\mathcal{F}_A} \frac{h(B,G)}{|G|\cdot|A:B|} \cdot x^{|A:B|}\right).$$

This theorem is a consequence of the following fact that is proved in [6] when $G = \{\varepsilon\}$ where ε is the identity element.

Let A be a finitely generated group, and let G be a finite group. Then,

$$\frac{h_n(A;G)}{|G|^n(n-1)!} = \sum_{|A:B|\leq n} \frac{h(B,G)}{|G|} \cdot \frac{h_{n-|A:B|}(A;G)}{|G|^{n-|A:B|}(n-|A:B|)!},$$

where the summation is over all subgroups B of A such that $|A:B|$ are less than n.

Sketch of proof. Let $G^{(n)}$ denote the direct product of n copies of G. Then S_n naturally acts on $G^{(n)}$, and the wreath product $G \wr S_n$ is the semidirect product of S_n and G. Define the action of $G \wr S_n$ on the set $G \times [n]$ where $[n] = \{1, 2, \ldots, n\}$ by

$$(g_1, g_2, \ldots, g_n)\sigma.(g,i) = (g_{\sigma(i)}g, \sigma(i)) \quad \in G \times [n],$$
$$(g_1, g_2, \ldots, g_n) \in G^{(n)}, \quad \sigma \in S_n, \quad (g,i) \in G \times [n].$$

This action is semiregular, and hence, $G \wr S_n$ is embedded in the symmetric group $S_{|G|n}$ on $|G|n$ letters. Let $\varphi \in \mathrm{Hom}(A, G \wr S_n)$. Let us define the following:

r is a positive integer such that $r \leq n$;
B is a subgroup of A such that $|A:B| = r$;
$\kappa \in \mathrm{Hom}(B, G)$;
π is a mapping from the set of all cosets of B in A to $[n]$ such that $\pi(B) = 1$;
$(y_1, y_2, \ldots, y_r)$ is an element of $G^{(r)}$ where $y_1 = \varepsilon$;
$\psi \in \mathrm{Hom}(A, G \wr S_{n-r})$.

Let μ_n be the homomorphism from $G \wr S_n$ to S_n defined by

$$\mu_n : G \wr S_n \ni (g_1, g_2, \ldots, g_n)\sigma \longrightarrow \sigma \in S_n.$$

Let B be the subset of A consisting of all elements a of A such that $\mu_n \circ \varphi(a)(1) = 1$, and let $r = |A:B|$. Define the homomorphism κ from B to G by

$$\varphi(b).(\varepsilon, 1) = (\kappa(b), 1)$$

where $b \in B$. Let $a_1B, a_2B, \ldots, a_rB$, where $a_1 = \varepsilon$, be all cosets of B in A, i.e.,

$$A = a_1B \cup a_2B \cup \cdots \cup a_rB.$$

Define an element $(y_1, y_2, \ldots, y_r)$ of $G^{(r)}$ and a mapping π from the set of all cosets $\{a_1B, a_2B, \ldots, a_rB\}$ to $[n]$ by

$$\varphi(a_j).(\varepsilon, 1) = (y_j, \pi(a_jB))$$

for each j. In particular, $y_1 = \varepsilon$. Let $\{k_1, k_2, \ldots, k_{n-r}\}$ where $k_1 < k_2 < \cdots < k_{n-r}$ be the subset of $[n]$ such that

$$[n] = \{\pi(a_1B), \pi(a_2B), \ldots, \pi(a_rB)\} \cup \{k_1, k_2, \ldots, k_{n-r}\}.$$

We define a homomorphism ν from $\varphi(A)$ to $G \wr S_{n-r}$ by

$$\nu : \varphi(A) \ni (g_1, g_2, \ldots, g_n)\sigma \longrightarrow (g_{k_1}, g_{k_2}, \ldots, g_{k_{n-r}})\sigma \in G \wr S_{n-r}.$$

Put $\psi = \nu \circ \varphi$. Thus, we get $r, B, \kappa, \pi, (y_1, y_2, \ldots, y_r)$ and ψ. Then, the correspondence

$$\varphi \longrightarrow \{r, B, \kappa, \pi, (y_1, y_2, \ldots, y_r), \psi\}$$

is a bijection. Therefore, we have that

$$h_n(A; G) = \sum_{|A:B| \leq n} h(B, G) \frac{(n-1)!}{(n - |A:B|)!} |G|^{|A:B|-1} h_{n-|A:B|}(A; G).$$

The result follows from this. $\square$

Suppose that A is a finitely generated ***abelian*** group. We denote by $\Phi_2(A)$ the intersection of all maximal subgroups of index 2 in A. The wreath product $G \wr A_n$ is a subgroup of $G \wr S_n$. For each subgroup C of A containing $\Phi_2(A)$, let

$$h_n^+(A:C;G) =^\sharp \{\varphi \in \mathrm{Hom}(A, G\wr S_n) | \varphi(C) \subset G \wr A_n\}.$$

In particular, $h_n^+(A:\Phi_2(A);G) = h_n(A;G)$, and $h_n^+(A:A;G) = h(A;G\wr A_n)$.

Define the subgroup $Al_n(G)$ of $G\wr S_n$ by

$$Al_n(G) = \{(g_1, g_2, \ldots, g_n)\sigma \in G\wr S_n | \mathrm{ord}_2(|g_1 g_2 \cdots g_n|) < \mathrm{ord}_2(|G|)\},$$

where $\mathrm{ord}_2(x)$ is the largest integer such that $2^{\mathrm{ord}_2(x)}$ divides x for each nonzero integer x. If 2 divides $|G|$ and if a Sylow 2-subgroup of G is not a cyclic group, then $Al_n(G) = G\wr S_n$. If 2 does not divide $|G|$, let $Al_n(G) = G \wr A_n$. As was mentioned earlier, $G\wr S_n$ is embedded in $S_{|G|n}$. Then, $Al_n(G)$ is identified with $A_{|G|n} \cap G \wr S_n$. Thus, $Al_n(C_2) = W(D_n)$ where $W(D_n)$ is the Weyl group. For each subgroup C of A containing $\Phi_2(A)$, let

$$h_n^-(A:C;G) =^\sharp \{\varphi \in \mathrm{Hom}(A, G\wr S_n) | \varphi(C) \subset Al_n(G)\}.$$

In particular, $h_n^+(A:\Phi_2(A);G) = h_n(A;G)$, and $h_n^+(A:A;C_2) = h(A;W(D_n))$.

We shall present the generating functions for $h_n^+(A:C;G)$ and $h_n^-(A:C;G)$. For each subgroup D of A containing $\Phi_2(A)$ and for each $B \in \mathcal{F}_A$, define

$$f_A^D(B,G) = \begin{cases} -h(B,G) & \text{if a Sylow 2-subgroup of } A/B \text{ is a cyclic group} \\ & \text{that is not } \{\varepsilon\} \text{ and that } A = DB, \\ h(B,G) & \text{otherwise,} \end{cases}$$

$$h_A^D(B,G) =^\sharp \{\kappa \in \mathrm{Hom}(B,G) | \mathrm{ord}_2(|\kappa(a^{|A:B|})|) = \mathrm{ord}_2(|G|) \text{ for some } a \in D\},$$

$$g_A^D(B,G) = \begin{cases} h(B,G) - 2h_A^D(B,G) & \text{if either a Sylow 2-subgroup of } |G| \text{ is} \\ & \text{a cyclic group that is not } \{\varepsilon\} \text{ or } B \in \mathcal{I}_A^D, \\ h(B,G) & \text{otherwise.} \end{cases}$$

For each subgroup D of A containing $\Phi_2(A)$, let

$$\begin{aligned} E_A^D(+,G;x) &= \exp\left(\sum_{B\in\mathcal{F}_A} \frac{f_A^D(B,G)}{|G|\cdot|A:B|}\cdot x^{|A:B|}\right), \\ E_A^D(-,G;x) &= \exp\left(\sum_{B\in\mathcal{F}_A} \frac{g_A^D(B,G)}{|G|\cdot|A:B|}\cdot x^{|A:B|}\right). \end{aligned}$$

It follows from definition that

$$f_A^{\Phi_2(A)}(B,G) = g_A^{\Phi_2(A)}(B,G) = h(B,G)$$

where $B \in \mathcal{F}_A$. Put $E_A(G;x) = E_A^{\Phi_2(A)}(+,G;x) = E_A^{\Phi_2(A)}(-,G;x)$.

Theorem 1. 2 ([4]) *Let A be a finitely generated abelian group and C a subgroup of A containing $\Phi_2(A)$. We denote by $\mathcal{K}_A^C$ the set of all subgroups D of C that contain $\Phi_2(A)$ as a subgroup of index 2. Let G be a finite group. Then,*

$$1+\sum_{n=1}^{\infty}\frac{h_n^+(A:C;G)}{|G|^n n!}\cdot x^n=\frac{1}{|C:\Phi_2(A)|}\left\{E_A(G;x)+\sum_{D\in\mathcal{K}_A^C}E_A^D(+,G;x)\right\}.$$

Suppose that a Sylow 2-subgroup of G is a cyclic group that is not $\{\varepsilon\}$. Then

$$1+\sum_{n=1}^{\infty}\frac{h_n^-(A:C;G)}{|G|^n n!}\cdot x^n=\frac{1}{|C:\Phi_2(A)|}\left\{E_A(G;x)+\sum_{D\in\mathcal{K}_A^C}E_A^D(-,G;x)\right\}.$$

Remark. In the paper [2], N. Chigira proved this theorem in the case where A is a cyclic group.

Example. Let $A=C_2^{(t)}$. For each subgroup B of A, $h(B,C_2)=|B|$, and hence

$$E_A(C_2;x)=\exp\left(\sum_{B\le A}\frac{|B|}{2|A:B|}\cdot x^{|A:B|}\right).$$

For each cyclic subgroup D of order 2 in A and for each subgroup B of A,

$$f_A^D(B,C_2)\;=\begin{cases}-|B| & \text{if } |B|=2^{t-1} \text{ and if } A=DB,\\ |B| & \text{otherwise,}\end{cases}$$

$$h_A^D(B,C_2)\;=\begin{cases}2^{t-1} & \text{if } B=A,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, for any cyclic subgroup D of order 2 in A,

$$\begin{aligned}E_A^D(+,C_2;x)&=E_A(C_2;x)\exp\left(-2^{2t-3}x^2\right),\\ E_A^D(-,C_2;x)&=E_A(C_2;x)\exp\left(-2^{t-1}x\right).\end{aligned}$$

Since the number of cyclic subgroups of order 2 in A is 2^t-1, it follows that

$$\begin{aligned}1+\sum_{n=1}^{\infty}\frac{h(C_2^{(t)},C_2\wr A_n)}{n!}\cdot x^n&=\frac{1}{2^t}E_{C_2^{(t)}}(C_2;x)\left\{1+(2^t-1)\exp\left(-2^{2t-3}x^2\right)\right\},\\ 1+\sum_{n=1}^{\infty}\frac{h(C_2^{(t)},W(D_n))}{2^n n!}\cdot x^n&=\frac{1}{2^t}E_{C_2^{(t)}}(C_2;x)\left\{1+(2^t-1)\exp\left(-2^{t-1}x\right)\right\}.\end{aligned}$$

2 The number of homomorphisms from a cyclic p-group to a symmetric group

Let A be a finite abelian group. It follows from [7] that

$$|\mathrm{Hom}(A, G)| \equiv 0 \bmod \gcd(|A|, |G|)$$

for any finite group G. This result is a generalization of the theorem of Frobenius:

$$\sharp\{x \in G | x^d = 1\} \equiv 0 \bmod \gcd(d, |G|).$$

Let $h_n(A) = |\mathrm{Hom}(A, S_n)|$. Let us study $\mathrm{ord}_p(h_n(A))$ where p is a prime integer. We denote by $m_A(d)$ the number of subgroups of index d in A. Put

$$E_A(x) = \exp\left(\sum_{d=1}^{|A|} \frac{m_A(d)}{d} \cdot x^d\right).$$

Then, it follows from Theorem 1.1 that

$$E_A(x) = 1 + \sum_{n=1}^{\infty} \frac{h_n(A)}{n!} \cdot x^n.$$

As a special case, we obtain

$$E_{C_{p^l}}(x) = \exp\left(\sum_{k=0}^{l} \frac{1}{p^k} \cdot x^{p^k}\right),$$

where C_{p^l} is a cyclic p-group of order p^l. The p-adic power series $E_p(x)$ is defined by

$$E_p(x) = \exp\left(\sum_{k=0}^{\infty} \frac{1}{p^k} \cdot x^{p^k}\right),$$

which is called the *Artin-Hasse exponential.* It is well known that $E_p(x) \in \mathbf{Z}_p[[x]]$, where $\mathbf{Z}_p$ is the ring of p-adic integers. Put $E_p(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, this fact yields that $\mathrm{ord}_p(a_n) \geq 0$ for any n. If $n < p^{l+1}$, then $a_n = h_n(C_{p^l})/n!$ and then

$$\mathrm{ord}_p(h_n(C_{p^l})) \geq \mathrm{ord}_p(n!),$$

where

$$\mathrm{ord}_p(n!) = \sum_{j=1}^{\infty} \left[\frac{n}{p^j}\right].$$

Furthermore, we have the following.

Theorem 2. 1 ([5]) *For each positive integer n,*

$$\mathrm{ord}_p(h_n(C_{p^l})) \geq \sum_{j=1}^{l} \left[\frac{n}{p^j}\right] - l\left[\frac{n}{p^{l+1}}\right],$$

and equality holds if $n \equiv 0 \bmod p^{l+1}$.

To prove this theorem, we use the decomposition:

$$E_{C_{p^{l+1}}}(x) = \exp\left(\frac{1}{p^{l+1}} \cdot x^{p^{l+1}}\right) E_{C_{p^l}}(x).$$

Example. For each positive integer n,

$$\mathrm{ord}_2(h_n(C_2)) = \begin{cases} \left[\frac{n}{2}\right] - \left[\frac{n}{4}\right] + 1 & \text{if } n \equiv 3 \bmod 4, \\ \left[\frac{n}{2}\right] - \left[\frac{n}{4}\right] & \text{otherwise.} \end{cases}$$

3 The number of subgroups of a finite abelian p-group

Let P be a finite abelian p-group. The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \lambda_{r+2} = \cdots = 0$$

is called the *type* of P if P is isomorphic to the direct product of cyclic groups

$$C_{p^{\lambda_1}} \times C_{p^{\lambda_2}} \times \cdots \times C_{p^{\lambda_r}}.$$

We write $|\lambda| = s$ if λ is the type of a finite abelian p-group of order p^s. For the partition λ, let $\alpha_\lambda(i;p)$ denote the number of subgroups of order p^i in a finite abelian p-group of type λ, which is a polynomial in p with nonnegative coefficients and depends only on λ and i. It is well known that $\alpha_\lambda(i;p) = \alpha_\lambda(s-i;p)$ if $|\lambda| = s$. It follows from [3] that for each partition λ, if $|\lambda| = s$, then

$$\alpha_\lambda(i;p) - \alpha_\lambda(i-1;p) = p^i \alpha_{\hat{\lambda}}(i;p) - p^{s-i+1}\alpha_{\hat{\lambda}}(s-i+1;p),$$

where $\hat{\lambda} = (\lambda_2, \ldots, \lambda_r, \ldots)$. Using this fact, we have the following theorem in [1]:

Let λ be a partition. Let $|\lambda| = s$ and $t = s - \lambda_1$. Then $\alpha_\lambda(i;p) - \alpha_\lambda(i-1;p)$ has nonnegative coefficients; moreover

$$\begin{aligned} &\alpha_\lambda(i;p) \equiv \alpha_\lambda(i-1;p) + p^i \bmod p^{i+1} && \textit{if } 0 \leq i \leq \min\left\{t, \left[\frac{s}{2}\right]\right\}, \\ &\alpha_\lambda(i;p) = \alpha_\lambda(i-1;p) && \textit{if } t < i \leq \left[\frac{s}{2}\right]. \end{aligned}$$

Example. Let $\lambda = (1,1,1,1,1,1)$ that is the type of $C_p^{(6)}$. Let $\alpha_\lambda(i;p) = \sum_{j=0} a_{i,j}p^j$. Then, we get the coefficients $a_{i,j}$ as follows.

	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	$a_{i,5}$	$a_{i,6}$	$a_{i,7}$	$a_{i,8}$	$a_{i,9}$
$\alpha_\lambda(0;p)$	**1**									
$\alpha_\lambda(1;p)$	1	**1**	1	1	1	1				
$\alpha_\lambda(2;p)$	1	1	**2**	2	3	2	2	1	1	
$\alpha_\lambda(3;p)$	1	1	2	**3**	3	3	3	2	1	1
$\alpha_\lambda(4;p)$	1	1	**2**	2	3	2	2	1	1	
$\alpha_\lambda(5;p)$	1	**1**	1	1	1	1				
$\alpha_\lambda(6;p)$	**1**									

4 A decomposition of $E_P(x)$

Let P be a finite abelian p-group of type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ where $|\lambda| = s$. Then $\alpha_\lambda(s-k;p) = m_P(p^k)$, and

$$E_P(x) = \exp\left(\sum_{k=0}^{s} \frac{\alpha_\lambda(s-k;p)}{p^k} x^{p^k}\right).$$

Let $\alpha_\lambda(i;p) = \sum_j a_{i,j}p^j$. Define the integers $l(\lambda)$ and $m(\lambda)$ by

$$\begin{aligned} l(\lambda) &= \max\left\{\lambda_1, \left[\frac{s+1}{2}\right]\right\}, \\ m(\lambda) &= s - l(\lambda). \end{aligned}$$

To simplify the notation, write $l = l(\lambda)$ and $m = m(\lambda)$.

Definition 4. 1 *For each pair (v,u) of nonnegative integers such that $v \le s$, let*

$$c_{u,v} = \begin{cases} b_{u,v} - b_{u-1,v} & \text{if } 0 \le v \le m \text{ and if } 0 \le u \le s-v, \\ a_{u,v} - a_{u-1,v} & \text{if } m < v \le s \text{ and if } 0 \le u \le s-v, \\ a_{v,u} & \text{if } s-v < u \end{cases}$$

where $b_{u,v} = a_{u,v} - a_{u-1,v-1}$.

We have a decomposition of the series $E_P(x)$ as follows.

Theorem 4. 1 ([5]) *For any u and v, $c_{u,v} \ge 0$, and*

$$E_P(x) = F_P(x) \cdot \prod_{v=0}^{s} \prod_{u=s-v+1}^{\infty} \exp(p^{u+v-s} x^{p^{s-v}})^{c_{u,v}},$$

$$F_P(x) = \left\{\prod_{v=0}^{m}\prod_{u=0}^{v} E_{C_{p^{l-u}} \times C_{p^{m-v}}}(x^{p^v})^{c_{u,v}}\right\} \cdot \left\{\prod_{v=m+1}^{s}\prod_{u=0}^{s-v} E_{C_{p^{s-u-v}}}(x^{p^v})^{c_{u,v}}\right\}.$$

In the proof of this theorem, we use the preceding results that relate to the number of subgroups.

Example. Let $P = C_p^{(6)}$. Then $l = m = 3$, and

$$F_P(x) = E_{C_{p^3}\times C_{p^3}}(x)E_{C_{p^2}\times C_p}(x^{p^2})E_{C_p\times C_p}(x^{p^2})E_{C_p}(x^{p^4})\exp(x^{p^4})\exp(x^{p^5}).$$

Remark. Let l be an integer, and let m be an integer such that $l \geq m$. Then,

$$\alpha_{(l,m)}(i;p) = \begin{cases} 1+p+\cdots+p^i & 0 \leq i < m, \\ 1+p+\cdots+p^m & m \leq i \leq l, \\ 1+p+\cdots+p^{l+m-i} & l < i \leq l+m. \end{cases}$$

Using this fact, we have that

$$E_{C_{p^l}\times C_{p^m}}(x) = E_{C_{p^{l+m}}}(x)E_{C_{p^{l+m-2}}}(x)\cdots E_{C_{p^{l-m}}}(x).$$

5 The number of homomorphisms from a finite abelian group to a symmetric group

Using Theorem 4.1, we have the following.

Theorem 5. 1 ([5]) *Let A be a finite abelian group such that the type of a Sylow p-subgroup of A is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ where $|\lambda| = s$. Let*

$$l(\lambda) = \max\left\{\lambda_1, \left[\frac{s+1}{2}\right]\right\}.$$

Then for each positive integer n,

$$\mathrm{ord}_p(h_n(A)) \geq \sum_{j=1}^{l(\lambda)}\left[\frac{n}{p^j}\right] - (2l-s)\left[\frac{n}{p^{l(\lambda)+1}}\right],$$

and the equality holds if $n \equiv 0 \bmod p^{l(\lambda)+1}$, except for the cases where $p = 2$ and $2l(\lambda) = s \geq 2$. Suppose that $p = 2$ and that $2l(\lambda) = s \geq 2$. Then for each positive integer n,

$$\mathrm{ord}_2(h_n(A)) \geq \sum_{j=1}^{l(\lambda)}\left[\frac{n}{p^j}\right] + \left[\frac{n}{2^{l(\lambda_1)+2}}\right] - \left[\frac{n}{2^{l(\lambda_1)+3}}\right],$$

and the equality holds if $n \equiv 0 \bmod 2^{l(\lambda)+1}$ and if $n \not\equiv 2^{l(\lambda)+1} + 2^{l(\lambda)+2} \bmod 2^{l(\lambda)+3}$.

Corollary 5. 1 ([5]) *Let A be a finite abelian group such that the type of a Sylow p-subgroup of A is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ where $|\lambda| = s$. Let*

$$l(\lambda) = \max\left\{\lambda_1, \left[\frac{s+1}{2}\right]\right\}.$$

Then the p-adic power series

$$E_A(x) = \sum_{n=0}^{\infty} \frac{h_n(A)}{n!} \cdot x^n$$

converges for

$$\mathrm{ord}_2(x) > \ 1 - \sum_{i=1}^{l(\lambda)} \frac{1}{2^i} - \frac{1}{2^{l(\lambda)+2}} + \frac{1}{2^{l(\lambda)+3}} \quad \textit{if } p = 2 \textit{ and if } 2l(\lambda) = s,$$

$$\mathrm{ord}_p(x) > \ \frac{1}{p-1} - \sum_{i=1}^{l(\lambda)} \frac{1}{p^i} + \frac{2l(\lambda) - s}{p^{l(\lambda)+1}} \qquad \textit{otherwise.}$$

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